Spin Foam Models of n-dimensional Quantum Gravity, and Non-Archimedan and Non-Commutative Formulations

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Abstract. This paper is twofold. First of all a complete unified picture of n-dimensional quantum gravity is proposed in the following sense: In spin foam models of quantum gravity the evaluation of spin networks play a very important role. These evaluations correspond to amplitudes which contribute in a state sum model of quantum gravity. In [6], the evaluation of spin networks as integrals over internal spaces was described. This evaluation was restricted to evaluations of spin networks in n-dimensional Euclidean quantum gravity. Here we propose that a similar method can be considered to include Lorentzian quantum gravity. We therefore describe the the evaluation of spin networks in the Lorentzian framework of spin foam models. We also include a limit of the Euclidean and Lorentzian spin foam models which we call Newtonian. This Newtonian limit was also considered in [10].

Secondly, we propose an alternative formulation of spin foam models of quantum gravity with its corresponding evaluation of spin networks. This alternative formulation is a non-archimedean or p-adic spin foam model. The interest on this description is that it is based on a discrete space-time, which is the expected situation we might have at the Planck length; this description might lead us to an alternative regularisation of quantum gravity. Moreover a non-commutative formulation follows from the non-archimedean one.

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1 Introduction

In [10], a definition for the evaluation of spin networks in 3-dimensional quantum gravity was considered. This was done in a unified spirit of three cases of quantum gravity which are the Lorentzian, the Euclidean and the Newtonian cases. [1] Particular attention was paid to the evaluation and asymptotics of the tetrahedron(3-simplex) network. Then it was suggested that the same work could be generalised by considering evaluations of spin networks in the n-dimensional case.

The evaluation of spin networks in the n-dimensional case was considered and restricted to the Euclidean case in [6]. This evaluation was described as Feynman graphs. These are integrals over internal spaces.

In this paper we generalise the work done in [10] and at the same time we complete in part the story of [6] by considering the evaluation of the n-dimensional Lorentzian and Newtonian n-simplex. We restrict ourselves to the case of the principal unitary irreducible representations of $SO(n-1,1)$ and $ISO(n-1)$.

In section 2 we summarise the results on n-dimensional Euclidean quantum gravity studied in [6] and give the recipe of the evaluations of spin networks as simple graphs. [2]

In section 3 we describe the case of n-dimensional Lorentzian quantum gravity and consider the case of spacelike spin networks, that is, graphs whose edges are all spacelike.

In section 4 we describe the limitng case of Newtonian spin networks, a limit of both, Euclidean and Lorentzian quantum gravity. This case should be interpreted as a mathematical idea which physically is given when the speed of light is so large. However, we doubt its physical interpretation of a Newtonian quantum gravity as this may not have sense at all as it will be addressed in section 4. Anyway, it is still interesting to have a physical interpretation of the Newtonian spin networks and to explore its impotance for spin foam models, if any.

The second part of the paper starts in section 5 where we propose a study of a p-adic and quantum deformation formulation of quantum gravity inspired on the spin foam models. There is a possible relation to a non-archimedean and non-commutative geometry respectively. This last section may be of great interest although future work is required to give it a precise formulation and to develop its possible importance. There is a thought that an adelic description of quantum gravity in terms of spin foams may be needed.

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[1] This latter case was given its name first in [10], and should be interpreted as a mathematical limit. This is because physically, there may not be sense for quantizing Newtonian gravity. This is better expained in section 4.

[2] A formulation of the existence of higher dimensional gravity theories appears in [7], therefore it is worth to study the evaluation of spin networks in any dimension.

2   Euclidean spin networks

A way to evaluate simple spin networks for the group $SO(n)$ was introduced in [6]. In this section we give a description of this idea which follows similarly for the Lorentz group $SO(n-1,1)$ and for the inhomogeneous group $ISO(n-1)$ which will be discussed in the following sections.

Simple spin networks of the group $SO(n)$ are those constructed from special representations of $SO(n)$. A special class of representations are the spherical harmonics [11] that appear in the decomposition in the decomposition of the space of fuctions $L^2(\mathbf{S}^{n-1})$ into irreducible components. These representations are labelled by integers ℓ and the spin networks, Σ, have edges labelled by ρ's. The evaluation of these spin networks is given by the following rules which closely resemble the rules of Feynman graphs that apear in quantum field theory:

-With every edge of the graph Σ, associate a propagator $K_\rho^E(x, y)$.

-Take the product of all these data and integrate over one copy of the homogeneous space $SO(n)/SO(n-1) = \mathbf{S}^{n-1}$ for each vertex.

Thus the evaluation formula is given by

$$\int_{\mathbf{S}^{n-1}} \prod_v dx_v \prod_e K_{\rho_e}^E(x, y) \tag{1}$$

where ρ_e denotes the representation labelling the edge e.

The propagators $K_\rho^E(x, y)$ are given by the Gegenbauer polynomials C_n^p as described in [6].

In [10] we discussed that the propagators in n-dimensional quantum gravity for any case(Euclidean, Lorentzian, Newtonian) are given by zonal spherical functions of the respective group. In this Euclidean case these propagators are Legendre polynomials and are directly related to the Gegenbauer polynomials [19].

These propagators are then expressed in an integral form as

$$K_\rho^E(r) = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}\Gamma(\frac{n-2}{2})} \int_0^\pi (\cos\theta + i\cos\psi\sin\theta)^\sigma \sin^{n-3}\psi d\psi \tag{2}$$

where $\sigma = -p + i\ell$. p is related to the dimension of the space-time as $p = (n-2)/2$ and ℓ labels the irreducible representations os $SO(n)$.

As a special case we have the n-simplex K^{n+1}, which is the complete graph with $n+1$ vertices.[3]

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[3]A complete graph K^{n+1} has $n+1$ vertices and an edge for each pair of vertices, so that it has $n(n+1)/2$ edges.
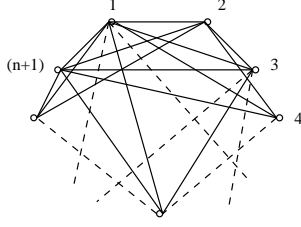
Its evaluation is given by the integral

$$K^{n+1} = \int_{\mathbf{S}^{n-1}} dx_1...dx_{n+1} \prod_e K^N_{\rho_e}(r(x,y)) \tag{3}$$

and its asymptotics which was studied in [6] and specialised to the 3-dimensional case in [10] is given by

$$\cos\Big(\sum_{i<j}(V_{ij}\theta_{ij} + k\frac{\pi}{4})\Big) \tag{4}$$

where V_{ij} are volumes of $(n-2)$-simpleces and k is an integer expressed in terms of the dimension n, [6]. In the 3-dimensional case we have that V_{ij} are lengths of edges. The formula was obtained in [10] and in that case, k is the Hessian of the action.

All these method can be generalised to include the Lorentzian case and then to the Newtonian limit. This is done in the following sections.

3   Lorentzian spin networks

For the n-dimensional Lorentzian case we have n-dimensional Minkowski space given by $\mathbf{R}^n$ with bilinear form

$$[x,y] = x_0y_0 - x_1y_1 - ... - x_{n-1}y_{n-1}$$

The unimodular group which leaves this bilinear form invariant is denoted by $SO(n-1,1)$. The light cone, also known by null cone, C is the set of points $x \in \mathbf{R}^n$ which satisfy $[x,x] = 0$.

The subgroup of $SO(n-1,1)$ of transformations preserving both sheets of the cone is the connected component denoted by $S_0(n-1,1)$. The language of representation theory is important in the spin foam models formulations of quantum gravity. For our group $S_0(n-1,1)$, its representations are mainly divided in a set of discrete representations labelled by integers and in a set of continuous representations labelled by real numbers.

In this paper we restrict ourselves to the continuos representations known as the principal unitary series. These representations are thought as labelling

spin networks with spacelike edges. A state sum model of quantum gravity is constructed by considering a triangulation $\triangle$ of an n-dimensional manifold M and considering its dual complex $\mathcal{J}_\triangle$, we construct a spin foam model by labelling each face of $\mathcal{J}_\triangle$ by a principal unitary irreducible representation of the group $SO_0(n-1,1)$. The state sum model is the given by

$$\mathcal{Z}(M) = \int_0^\infty d\rho_f \prod_f A(f) \prod_e A(e) \prod_v A(v) \tag{5}$$

where the integration is carried over the labels of all internal faces of the dual complex and $A(f), A(e), A(v)$ are the amplitudes given to the faces, edges and vertives of the dual complex $\mathcal{J}_\triangle$. These amplitudes are given by the evaluation of certain labelled spin networks such as

$$A(f) = \quad \bullet \!\!\bigcirc\!\! \quad \rho$$

$$A(e) = 1/ \quad \text{n edges}$$

$$A(v) =$$

Recall that the egdes of these graphs are labelled by principal unitary representations of $SO_0(n-1,1)$. The vertex amplitude is given by the evaluation of the n-simplex graph. This n-simplex can be seen as a collection of $n+1$ points which are all joined together by edges.

The evaluation of these graphs is analogous to the Euclidean case, so the recipe is:

-To each edge of the spin network we associate a propagator. The propagators are given by zonal spherical funtions for the continious representations of $SO_0(n-1,1)$.

-We multiply all these propagators and integrate over a copy of an internal space for each vertex. The internal spaces are homogeneous spaces in which the propagators are evaluated.

For the group $SO_0(n-1,1)$ the zonal spherical functions for the continuos represenations are given by the Legendre functions

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$$K_\rho^L(r) = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}\Gamma(\frac{n-2}{2})} \int_0^\pi (\cosh r - \cos\theta \sinh r)^\sigma \sin^{n-3}\theta d\theta \qquad (6)$$

where $\sigma = -p + i\rho$. p is related to the dimension of the space-time as $p = (n-2)/2$, and ρ labels a continuous representation of $SO_0(n-1,1)$.

Moreover, the homogeneous space is given by the $(n-1)$-dimensional hyperbolic space $\mathbf{H}_\infty^{n-1} = SO_0(n-1,1)/SO(n-1)$. [4]

3.1 The n-simplex

The non-degenerate n-simplex is represented as the complete graph of $(n+1)$ vertices denoted by K^{n+1}. Following the recipe for evaluation of spin networks, the evaluation of the n-simplex spin network graph would be given by the following integral

$$K^n = \int_{\mathbf{H}_\infty} dx_1...dx_{n-1} \prod_e K_{\rho_e}^L(r(x,y)) \qquad (7)$$

where we have a multiple integral over n vertices of the n-simplex. One of the integrals was dropped for regularisation in analogy to the 3-dimensional case [10], and the 4-dimensional case [3]. A problem that appears now is whether our integral (7) converges, that is, whether it is well defined. This problem also arised in the 3-dimensional case [10], and in the 4-dimensional case [3]. Here we give much evidence for its convergence in any dimension. This evidence is very close to a proof.

3.1.1 The convergence

Before showing the evidence for the convergence of the n-simplex in any dimension we first consider some other integral evaluations and show their convergence. Although this discussion follows closely the same ideas of the 4-dimensional case [1], it gives a generalisation to the n-dimensional case.

We have that for $\rho \neq 0$ our kernel $K_\rho^L(r)$ is well defined for small r and for large values of r is asymptotic to [15]

$$K_\rho^L(r) \sim A_\rho 2^{(n-3)/2}\Gamma(\frac{n-1}{2})e^{(1-\frac{n}{2})r} \qquad (8)$$

where n is the dimension of the space-time and A_ρ is a factor which depends on the representation.

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[4]Let us denote the $(n-1)$-dimensional hyperbolic space simply by $\mathbf{H}_\infty$. This notation will be clear when we study the non-archimedean formulation.

The theta symbol and more general graphs with loops: We prove first that the theta symbol converges in any dimension, that is the following evaluation converges

$$\langle\!\!\!\!\!\text{graph}\!\!\!\!\!\rangle = \int_{\mathbf{H}_\infty} dx\ K_{\rho_1}(x,y)K_{\rho_2}(x,y)K_{\rho_3}(x,y) \tag{9}$$

This follows easily from the asymptotic behaviour of our kernel where our integral can be written in the form

$$\sim A_{\rho_1}A_{\rho_2}A_{\rho_3}2^{3(n-3)/2}\Big(\Gamma(\frac{n-1}{2})\Big)^3\int_{\mathbf{H}_\infty} dx\ e^{(1-n/2)r}e^{(1-n/2)r}e^{(1-n/2)r} \tag{10}$$

The volume form gives

$$\sim A_{\rho_1}A_{\rho_2}A_{\rho_3}2^{3(n-3)/2}\Big(\Gamma(\frac{n-1}{2})\Big)^3\int^{\infty} dr\ e^{\frac{(2-n)}{2}r} \tag{11}$$

which for $n \geq 3$ is finite. This implies that the theta symbol converges in any dimension greater than or equal to 3.

Similarly, if we consider the evaluation of the edge amplitude A_e^{-1}, or any such graph of two vertices joined by $k \geq 3$ edges, we have that its evaluation is given by

$$\langle\!\!\!\!\!\text{graph}\!\!\!\!\!\rangle \sim A_{\rho_1}A_{\rho_2}...A_{\rho_k}2^{k(n-3)/2}\Big(\Gamma(\frac{n-1}{2})\Big)^k\int^{\infty} dr\ e^{k(1-n/2)r}e^{(n-2)r}$$

$$\sim A_{\rho_1}A_{\rho_2}...A_{\rho_k}2^{k(n-3)/2}\Big(\Gamma(\frac{n-1}{2})\Big)^k\int^{\infty} dr\ e^{\frac{(k-2)(2-n)}{2}r} \tag{12}$$

which is finite.

More general integrals: Now we consider the following integral

$$I = \int_{\mathbf{H}_\infty} dx\ K_{\rho_1}(x,x_1)K_{\rho_2}(x,x_2)...K_{\rho_k}(x,x_k) \tag{13}$$

for fixed $x_1,...,x_k \in \mathbf{H}_\infty$. We prove that it converges. Using the fact that our kernel $K_\rho(x,y)$ is asymptotic to $A_\rho 2^{(n-3)/2}\Gamma(\frac{n-1}{2})e^{(1-\frac{n}{2})r}$ we have that our integral I is asymptotic up to some factors, to

$$\int_{\mathbf{H}_\infty} dx\ e^{(1-\frac{n}{2})(r_1+...r_k)} \tag{14}$$

where $r_i = d(x, x_i)$. In [1] it was proved that we can find a barycentre $b \in H^+$ for the points x_i, such that [5]

$$r := d(x, b) \leq \frac{1}{k}(r_1 + ... + r_k) \tag{15}$$

Then in spherical coordinates around b we see that our integral I is bounded by

$$\int^\infty e^{k(1-\frac{n}{2})r} \sinh^{(n-2)} r \ dr \tag{16}$$

which converges for all $n \geq 3$ and all $k \geq 3$.

Consider again the integral (14). We have that

$$I \sim \int^\infty \sinh^{(n-2)} r e^{(1-\frac{n}{2})(r_1+...r_k)} \ dr \tag{17}$$

Now we proceed as in [1] by breaking the integral over r into two parts by considering:

$$r_1 + ... + r_k \geq kr \ , \ M = \frac{1}{k} min_x (r_1 + ... + r_k)$$

therefore we have that

$$I \sim \int^M e^{(1-\frac{n}{2})kM} \sinh^{(n-2)} r \ dr + \int_M^\infty e^{(1-\frac{n}{2})kr} \sinh^{(n-2)} r \ dr$$

$$\sim \frac{1}{2} \int^M e^{(1-\frac{n}{2})kM+(n-2)r} \ dr + \frac{1}{2} \int_M^\infty e^{(1-\frac{n}{2})kr+(n-2)r} \ dr$$

$$\leq B e^{(1-\frac{n}{2})kM+(n-2)M} \tag{18}$$

where B is a constant factor. The triangle inequality implies that

$$r_1 + ...r_k \geq \frac{1}{k-1} \sum_{i<j} r_{ij} \tag{19}$$

for all $x_1, .., x_k$ where $r_{ij} = d(x_i, x_j)$. Therefore we have that

$$M \geq \frac{1}{k(k-1)} \sum_{i<j} r_{ij} \tag{20}$$

which implies that

$$I \sim B \exp\left[((1-\frac{n}{2})k + (n-2))\frac{1}{k(k-1)} \sum_{i<j} r_{ij})\right] \tag{21}$$

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[5] Although in [1] the case of the 3-dimensional hyperbolic space was considered, the proof for the existence of a barycentre works in any dimension

Now we can go back to the n-simplex and compute for instance the integral over x_n. This resembles an integral of the same kind as the integral I. Therefore it is asymptotic to the equation (21) and we continue to integrate over the other variables. At some step we will have to change to spheroidal coordinates as was done in [1] for the 4-dimensional case. Following the procedure it can be expected that the integral that defines the evaluation of the n-simplex is bounded and therefore converges.

3.2 Asymptotics

Once we have shown evidence for the convergence of the amplitude evaluation of the n-simplex, we calculate its asymptotics.[6] First we notice that evaluating the integral of our kernel $K_\rho^L(r)$, we have that

$$K_\rho^L(r) = \frac{2^{(n-3)/2}\Gamma((n-1)/2)}{\sinh^{(n-3)/2} r}\mathfrak{B}_{\sigma+(n-3)/2}^{(3-n)/2}(\cosh r) \tag{22}$$

where $\mathfrak{B}_{\sigma+(n-3)/2}^{(3-n)/2}(\cosh r)$ are Legendre functions. We have that $\sigma = -p + i\rho$ where $p = (n-2)/2$ so that

$$\mathfrak{B}_{\sigma+(n-3)/2}^{(3-n)/2}(\cosh r) = \mathfrak{B}_{-(1/2)+i\rho}^{(3-n)/2}(\cosh r) \tag{23}$$

For large value of ρ we have that our kernel can be approximated by the asymptotic behaviour of $\mathfrak{B}_{-(1/2)+i\rho}^{(3-n)/2}(\cosh r)$ which is given by

$$\sqrt{\frac{2}{\sinh r}}\rho^{(2-n)/2}\cos\left(\rho r + \frac{(3-n)\pi}{4} - \frac{\pi}{4}\right) \tag{24}$$

We now have that our n-simplex evaluation can be written as

$$K^{n+1} = \int_{\mathbf{H}_\infty} dx_1...dx_{n-1}\prod_{i<j} e^{i\sum_{i<j}\epsilon_{ij}\rho_{ij}r_{ij}+(2-n)\pi/4} \tag{25}$$

The function given by $S = \sum_{i<j}\epsilon_{ij}\rho_{ij}r_{ij} + (2-n)\pi/4$ is called the action.

We rescale all of our representations ρ_{ij} by a common factor $\alpha\rho_{ij}$ and look for the behaviour of our integral formula when $\alpha \to \infty$. We use the stationary phase method to evaluate the asymptotics of our integral formula (25). Moreover we restrict to the non-degenerate configurations where all $r_{ij} \neq 0$.

Given our non-degenerate n-simplex, there are $(n+1)$ timelike unit vectors n_i which are normals to the $(n-1)-$simpleces. There is a notion of Lorentzian angle, from which a Schläfli identity follows, [1]. This identity is given by

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[6]The study of the asymptotics of amplitudes of spin networks in quantum gravity can be found in many other references such as [2], [4], [8], [10]

$$\sum_{i \leq j} V_{ij} d\Theta_{ij} = 0 \qquad (26)$$

Some of our normal vectors are future pointing and some are past pointing. We know that our points x_i live on the hyperbolic hyperspace $\mathbf{H}_\infty$ which implies that these vectors are all timelike and future pointing. Then $x_i = a_i n_i$ where $a_i = 1$ if n_i is future pointing, or $a_i = -1$ if n_i is past pointing.

By taking this into account we then vary the action and constrain such variation by a Lagrange multiplier which finally fixes the ϵ's up to an overall sign. We will then have an asymptotic behaviour given by an oscillatory function analogous to the Euclidean one.

4   Newtonian spin networks

This case of Newtonian spin networks has a mathematical sense, as it is a limit of the Euclidean and Lorentzian cases. However, its physical interpretation as a Newtonian quantum gravity theory is not clear to the author.

First of all, as it is mentioned in [12], quantum gravity refers to the attempts to unify general relativity and quantum theory. If gravity was nothing but the Newtonian well known static force, the constuction of a corresponding quantum theory would be a simple and uninteresting affair.

But even more, it may not have sense at all. This is because of the following reasoning: The non-degenerate scalar product which gives rise to the inhomogeneous symmetry of space-time is interpreted physically as a limiting case of the Lorentzian one when the speed of light $c \to \infty$. This is also correct form the Newtonian theory point of view where the principle of the universal speed limit given by the velocity of ligth c is no longer true. In Newtonian theory a body can reach any speed and so light could travel so fast in a reference frame. Therefore c is not a constant any more and the Planck length

$$\ell_P = \left(\frac{Gh}{c^3} \right)^{\frac{1}{2}}$$

has no meaning at all as a constant. If light $c \to \infty$, the Planck length $L_P \to 0$, and so, space-time is classical.

It is also possible that the name of Newtonian spin networks is not appropriate. Any way, the inhomogeneous limit has sense and we have the right to study the evaluation of its spin networks.

Mathematically the inhomogeneous group $ISO(n-1)$ can be obtained by a limit procedure from $SO_0(n-1,1)$ [19].

The Newtonian spin networks for n-dimensional quantum gravity whose group is $ISO(n-1)$, and whose bilinear form is degenerate should then be evaluated by a zonal spherical function of this group. That is,

$$K_\rho^N(r) = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}\Gamma(\frac{n-2}{2})} \int_0^\pi e^{r\rho\cos\theta} \sin^{n-3}\theta d\theta \qquad (27)$$

where as for the Lorentzian case, we also have a continuous set of representations denoted by ρ.

Moreover, the homogeneous space is given by $\mathbf{R}^{n-1} = ISO(n-1)/SO(n-1)$.

Our spherical functions (27) are Bessel functions of the corresponding dimensions.

Therefore the evaluation of the n-simplex in the Newtonian framework is given by the integral

$$K^{n+1} = \int_{\mathbf{R}^{n-1}} dx_1...dx_{n-1} \prod_e K_{\rho_e}^N(r(x,y)) \qquad (28)$$

This integral seems to be well defined since for small values of r, that is when $r \to 0$ the Bessel functions approach

$$\frac{r^{n-3}}{2^{(n-3)}\Gamma(1+(n-3))} \qquad (29)$$

moreover, the asymptotics of these functions for $\rho \to \infty$ is given by

$$\sqrt{\frac{2}{\pi\rho r}} \cos\left(\rho r - \frac{(n-3)\pi}{2} - \frac{\pi}{4}\right) \qquad (30)$$

As our Lorentzian case approaches the Newtonian case for small values of r, we can hope that our Newtonian tetrahedron evaluation is finite as in our Lorentzian case.

If the evaluation of the n-simplex converges by a similar argument to the Lorentzian one, then one must expect an oscillatory asymptotic behaviour.

5 Non-Archimedean and Non-Commutative Formulations

In this section we define two new formulations of possible spin foam models of quantum gravity which may be related to a non-archimedean and non-commutative property of space-time. These two models might be related to each other as it is commented in following subsections. There is a new revolution of our concept of space-time coming from quantum gravity, which is the idea that space-time is discrete at the Planck length. Moreover, it is believed that algebra and combinatorics play an important role for quantum gravity and that the concept of point in space-time is non-sense. Therefore a non-archimedean formulation includes the idea of a discreteness of space-time, and

a non-commutativity includes the idea of representing the whole information of a quantum space-time algebraically.

We develop this ideas here, and a more precise definition of these models require more study.

5.1 Non-Archimedean Spin Foam Models

We now propose the construction of a non-archimedean spin foam model of quantum gravity.[7] We see this new non-archimedean model as an alternative way of studying covariant quantum gravity and the most interesting situation about this non-archimedean model is that it might be an alternate way of renormalisation of quantum gravity in the physical context of state sum models. Also, it gives a discreteness of space-time which is expected at the fundamental level.

It is indeed a p-adic gravity constructed over a p-adic space-time. The field of real numbers $\mathbf{R}$ is an extension of the rationals $\mathbf{Q}$, but there are many other infinite extensions of the rationals with an equal right. These extensions are constructed for any prime number p, and are known by the name of p-adic numbers $\mathbf{Q}_p$, [14], [18].

As quantum field theory and string theory have their p-adic alternative formulation [18], [21], we propose that spin foam models of quantum gravity have a p-adic formulation over a p-adic space-time $\mathbf{Q}_p^n$.

For instance, in the case were $n = 3$ we have that the real 3-dimensional Lorentzian spin foam model of quantum gravity and its evaluation of spin networks as studied in [10], has an alternative formulation as a Lorentzian p-adic spin foam model.

Its corresponding spin networks should be evaluated with the help of harmonic analysis over the corresponding homogeneous spaces. These homogeneous spaces are the counterpart of the hyperbolic plane given by $SL(2,\mathbf{R})/SO(2)$.

The corresponding p-adic homogeneous spaces are given by $PGL(2,\mathbf{Q}_p)/PGL(2,\mathbf{Z}_p)$ [21]. These spaces are visualised as infinite trees with vertices of valance $p+1$. These trees are known to be lattice hyperbolic spaces which are called Bruhat-Tits trees(see figure 1).

We denote these spaces by $\mathbf{H}_p$. In this language the classical hyperbolic plane is denoted by $\mathbf{H}_\infty$. This means that we recover our previous notion of archimedean spin foam models of quantum gravity.

There is a notion of distance and of geodesics between vertices in these p-adic hyperbolic spaces $\mathbf{H}_p$. As our p-adic hyperbolic space is a tree, there is a unique path of edges between any two vertices. This unique path is considered to be a geodesic between the vertices, and the number of edges that form this path is the distance between these vertices. Then we can choose any vertex to be the centre of our space.
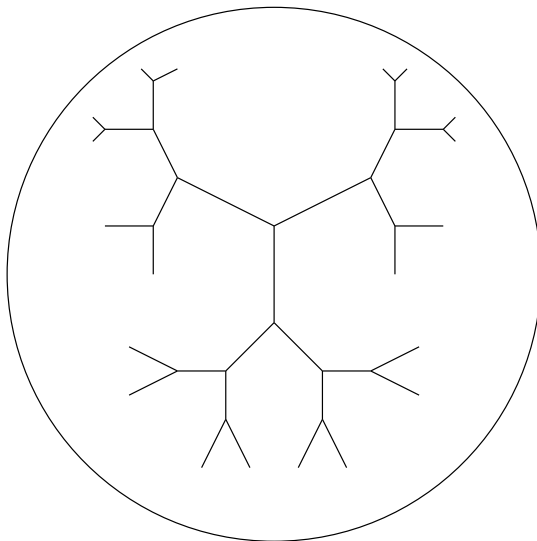
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[7]It will also be called p-adic spin foam model

Figure 1: Bruhat-Tits tree as a p-adic hyperbolic space. This particular figure shows an example of a 2-adic hyperbolic space

We know that in order to evaluate the amplitudes of spin networks, we need to do some harmonic analysis on our homogeneous spaces and find the corresponding zonal spherical functions. Fortunately, in [5], we can find a way to do harmonic analysis on trees. In [9], a study of zonal spherical functions over trees is found as well as the expression for the p-adic discrete laplacian. Moreover, the asymptotic behaviour of the p-adic zonal spherical functions for large distances r on the p-adic hyperbolic space is given by

$$\phi_\lambda^p \sim A_\lambda \; p^{-\frac{r}{2}} \tag{31}$$

were A_λ is a factor that depends on a real parameter λ. These λ's should be thought as labelling edges of p-adic spin networks.

Evaluation of p-adic spin networks We then give the recipe for the evaluation of p-adic spin networks, a recipe who resembles the one of spin networks of archimedean spin foam models of quantum gravity, a one that we already know very well.

Given a spin network graph whose edges are labelled by numbers λ, we define its evaluation as follows:

- To each edge of the spin network we associate a p-adic propagator. This p-adic propagator is given by the p-zonal spherical functions ϕ_λ^p.

-We multiply all these propagators and integrate over a copy of an p-adic internal space for each vertex. These p-adic internal spaces are given by the p-adic hyperbolic spaces $\mathbf{H}_p$, also known as Bruhat-Tits trees.

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It is interesting to study the possible convergence of these p-adic spin foam models and propose them as a possible regularisation of quantum gravity. These models are seen as an alternative description of quantum gravity which in a way also resembles a kind of lattice formulation because of the Bruhat-Tits trees which play an important role in the formulation. In this way we can see the p-adic formulation more than an alternative formulation, but as a generalization of spin foam models which is based on a discrete space-time; more like in lattice gauge theories. We should be able to obtain the continuous picture of our space-time, which suggests a possible adelic formulation of quantum gravity.

We think that all these ideas require a formal formulation and careful study and leave them to future work.

Another very related formulation to the p-adic one is given below and it is seen as a non-commutative one.

5.2 Non-Commutative Formulations

Nowadays there has been a deep relation between quantum field theory and noncommutative geometry. There are many issues of the 20th century physics which have been studied in the noncommutative framework.

In this section we propose a study of quantum gravity, based on a non-commutativity of space-time based on spin foam models. This implies that we should have a noncommutative counterpart of the classical BF action and of the constrained one which gives gravity. Moreover, the Feynman path integrals can be deformed to a noncommutative framework. This implies that they should correspond to spin foam models of the noncommutative BF and gravity theories.

This non-commutative picture is related in a way to the non-archimedean one, but it is more general.

We suppose that there is such way to obtain a spin foam theory of a non-commutative gravity theory and propose a way to evaluate the corresponding spin networks. These spin networks are completely abstract as in general our quantum spaces do not have points. We called such networks by the name of q-spin networks.

q-spin networks

Given a state sum model of non-commutative quantum gravity, we need to know how to evaluate the amplitudes of the vertices, edges and faces networks. We propose that the evaluation is given analogously to the case of spin networks of quantum gravity with vanishing cosmological constant by introducing a q-parameter such that $0 < q < 1$.

We define the evaluation of q spin networks in general. We now suppose we have a closed spin network and define its amplitude as a Feynman integral which evaluation rules are given as follows:

-To each edge of the spin network we associate a q-propagator. This q-propagator is given by q-zonal spherical functions of the representations of a quantum group G_q. These q-spherical functions are given by eigenfunctions of the q-Laplace-Beltrami operator which is the Casimir element of our quantum group.

-We multiply all these propagators and integrate over a copy of an q-internal space for each vertex. These q-internal spaces are analogous to the homogeneous spaces in which the propagators are evaluated. For the three cases of Euclidean, Lorentzian and Newtonian quantum gravity, these q-internal spaces are q-spheres(quantum spheres), q-hyperbolic spaces and q-planes(quantum planes).

For the Lorentzian case, there is a relation between the p-adic zonal spherical functions and the q-zonal sphercal functions [9]. This relation could lead us to a better understanding between the p-adic case and the non-commutative one. Moreover, this would lead us finally to a connection with the clasical case. We do not study any of this relations in the present paper but propose them as future problems.

It is also interesting to notice that we have another important similarity between the classical and quantum spherical functions, implying a possible way to formulate a meaning of non-commutative spin foam models in a formal way. For instance, in the case of Euclidean quantum gravity in which the lie algebra is given by $\mathbf{sl}_2$ the propagators which are zonal spherical functions(Legendre polynomials), are solutions of the well known Knizhnik-Zamolodchikov equations [13]. In fact these functions are special cases of the more general Gauss hypergeometric function given by

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k \tag{32}$$

which has an integral representation given by

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \tag{33}$$

Moreover, all solutions of the Knizhnik-Zamolodchikov equations are expressed in terms of generalized hypergeometric functions.

For the q-spherical special functions(little q-Legendre polynomials) we also have that such polynomials have similar properties as the usual Legendre polynomials. They are also orthogonal polynomials on the interval $[0, 1]$ with respect to a measure known as q-beta measure. They are also eigenfunctions of a second order q-differential operator. Moreover, they are solutions of the q-deformed versions of the Knizhnik-Zamolodchikov equations which in fact are special cases of the more general q-deformed versions of the Gauss hypergeometric functions given by

$$_2\phi_1(q^a, q^b; q^c; q, z) = \sum_{k=0}^{\infty}\Big(\prod_{j=0}^{k-1}\frac{[a+j][b+j]}{[c+j][1+j]}\Big)z^k \tag{34}$$

which has an integral representation given by

$$_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 t^{b-1}\frac{[(1-tz)^{-a}]}{[(1-t)^{b-c}]}\frac{d_q t}{1-t} \tag{35}$$

where this integral is a Jackson integral.

There is an extremely close relationship between these hypergeometric functions and representation theory of Kac-Moody Lie algebras and quantum groups.

Similarly we can generalise all this to the Lorentzian and Newtonian cases, where we have q-Legendre functions and q-Bessel functions respectively.

For instance there are two types of q-Bessel functions in equall foot as generalising the classical Bessel function. Their expressions are given by [20]

$$J_\nu^1(r; q) = \frac{1}{(q;q)_\nu}\Big(\frac{r}{2}\Big)^\nu {}_2\phi_1\Big(0, 0; q^{\nu+1}; q, -\frac{r^2}{4}\Big) \tag{36}$$

$$J_\nu^2(r; q) = \frac{1}{(q;q)_\nu}\Big(\frac{r}{2}\Big)^\nu {}_0\phi_1\Big(q^{\nu+1}; q, -\frac{r^2 q^{\nu+1}}{4}\Big) \tag{37}$$

Both these expressions are related by a factor and then up to these factor we can think of them as our propagators.

6 Conclusions

We gave much evidence for the possible convergence of the n-simplex in n-dimensional Lorentzian quantum gravity. We think that a similar reasoning would follow in the limit of Newtonian spin networks. This Newtonian case may need a good physical interpretation, as well as a argument of its importance to spin foasm models of quantum gravity. In [2], the study of degenerate configurations of the $10 - j$ symbol in Euclidean and Lorentzian 4-dimensional quantum gravity, led the author to conjecture that its evaluation is related to the evaluation of the so called degenerate spin networks which in our language are given by Newtonian spin networks. It is then interesting to understand better this limit of Newtonian spin networks and of its importance to spin foam models.

We have also proposed that a non-archimedean and non-commutative formulations of quantum gravity in terms of spin foam models may be possible. There is however much work to do, so that a formalisation to these ideas should be given.

We could try to describe both theories in terms of partition functions of a kind of constrained BF-theory. The corresponding lagrangians may be p-adic and non-commutative respectively.

The thought is that these models may give a regularisation of quantum gravity and that in the p-adic case, a discreteness is already implicit in the space-time. Even more, an adelic continuation of this formulation may be interesting which may include the already known spin foam models.

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